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One-Sided L^1 Approximation by Splines with Fixed Knots

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We prove that the best one-sided L^1 approximation by splines with fixed knots to a differentiable function is unique. Uniqueness need not hold for continuous functions.

1. INTRODUCTION

The purpose of this note is to indicate how the methods of Galkin [2], and Bojanic and DeVore [1] may be extended to a consideration of the problem of one-sided L^1 approximation by splines with fixed knots. This paper also supplements the cases considered by Micchelli and Pinkus [7], and Micchelli [6], where the function which was approximated, both in the one- and two-sided sense, was contained in the generalized convex cone defined with respect to the given class of splines with fixed knots. Our interest in this problem was further stimulated by the work of Meir and Sharma [5] and Ziegler [8], who considered one-sided approximations by splines in their study of "goodness of fit" to $f \in C^k[0, 1]$.

Let $\mathscr{S}_{n,r} = \{S(t)\colon S(t) = \sum_{i=0}^{n-1} a_i t^i + \sum_{i=1}^{r} c_i(t - \xi_i)_+^{n-1}\}$, where $0 < \xi_1 < \cdots < \xi_r < 1$ are fixed, denote the class of splines of degree $n - 1$ (order n) with r fixed knots. We consider the question of one-sided L^1 approximation to functions $f \in C[0, 1]$ by $S \in \mathscr{S}_{n,r}$, $n \geqslant 3$. Existence is readily shown and we shall concern ourselves with the question of uniqueness and characterization of the $S^* \in \mathscr{S}_{n,r}$, $S^*(t) \leqslant f(t)$, $t \in [0, 1]$, which satisfies

$$\int_0^1 (f(t) - S^*(t))\, w(t)\, dt \leqslant \int_0^1 (f(t) - S(t))\, w(t)\, dt,$$

for all $S \in \mathscr{S}_{n,r}$ for which $S(t) \leqslant f(t)$, $t \in [0, 1]$, where $w(t)$ is a positive weight function.

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In Section 2, we show that uniqueness is lacking for $f \in C[0, 1]$, while in Section 3 we prove that if $f \in C^1[0, 1]$, then there is a unique $S \in \mathscr{S}_{n,r}$ of best one-sided L^1 approximation to f from above and below.

2. Nonuniqueness for $f \in C[0,1]$

The following result is an immediate consequence of the work of Bojanic and DeVore [1].

Proposition. *There exists $f \in C[0, 1]$ for which there is not a unique spline in $\mathscr{S}_{n,r}$ of best one-sided L^1 approximation.*

Proof. From [6], there exists a quadrature formula (unique) of the form

$$\int_0^1 f(t)\, w(t)\, dt = \sum_{i=1}^{\ell} a_i f(t_i), \tag{1}$$

where $w(t)$ is as above, $a_i > 0, i = 1,..., \ell, 0 < t_1 < \cdots < t_\ell \leqslant 1$, $\ell = [(n + r + 1)/2]$ (if $n + r$ is even, all points are in the interior, and if $n + r$ is odd, $t_\ell = 1$), and which is exact for all $\mathscr{S}_{n,r}$.

Let $\tilde{S} \in \mathscr{S}_{n,r}$ be any nontrivial spline which vanishes at the $\{t_i\}_{i=1}^{\ell}$. Such splines exist. From (1),

$$\int_0^1 \tilde{S}(t)\, w(t)\, dt = 0. \tag{2}$$

Let

$$\begin{aligned} \tilde{S}(t)_+ &= \tilde{S}(t), \qquad && \tilde{S}(t) \geqslant 0, \\ &= 0, && \tilde{S}(t) < 0. \end{aligned}$$

Since $\tilde{S}(t)$ is nontrivial, it follows from (2) that

$$\int_0^1 \tilde{S}(t)_+\, w(t)\, dt > 0.$$

If $S \in \mathscr{S}_{n,r}$, $S(t) \leqslant \tilde{S}(t)_+$ for all $t \in [0, 1]$, then

$$\int_0^1 S(t)\, w(t)\, dt = \sum_{i=1}^{\ell} a_i S(t_i) \leqslant \sum_{i=1}^{\ell} a_i \tilde{S}(t_i)_+ = 0.$$

However, $\lambda\tilde{S}(t)$ satisfies

$$\int_0^1 \lambda\tilde{S}(t)\, w(t)\, dt = 0 \qquad \text{for} \quad 0 \leqslant \lambda \leqslant 1,$$

and $\lambda\tilde{S}(t) \leqslant \tilde{S}(t)_+$, $t \in [0, 1]$. The proposition follows. Q.E.D.

3. Uniqueness for $f \in C^1[0,1]$

Our main aim is to prove the following result.

THEOREM. *If $f \in C^1[0, 1]$, then the best one-sided L^1 approximation to $f(t)$ from $S \in \mathscr{S}_{n,r}$ is unique.*

Proof. Assume $S_1, S_2 \in \mathscr{S}_{n,r}$ are two best one-sided L^1 approximations to $f \in C^1[0, 1]$ from below. Due to the nature of splines, $S_1(t) \equiv S_2(t)$ on some set $V = [0, 1] \setminus \bigcup_{i=0}^{p} [\xi_{k_{2i}}, \xi_{k_{2i+1}}]$, and $S_1(t) \neq S_2(t)$ a.e. on $F = \bigcup_{i=0}^{p} [\xi_{k_{2i}}, \xi_{k_{2i+1}}]$, where $\xi_0 = 0$, $\xi_{r+1} = 1$, and $k_i \in \{0, 1,..., r + 1\}$, $0 \leqslant k_0 < \cdots < k_{2p+1} \leqslant r + 1$.

Since the set of best approximations is convex, it follows that there exist best approximations $S_1{}^*(t)$ and $S_2{}^*(t)$ which satisfy

(1) $f(t) - S_1{}^*(t) \neq 0$ a.e. on F;

(2) $f(t) - S_2{}^*(t) \neq 0$ a.e. on F;

(3) $S_1{}^*(t) - S_2{}^*(t) \neq 0$ a.e. on F;

(4) $S_1{}^*(t) \equiv S_2{}^*(t)$, $t \in V$.

This result is obtained by defining $S_1{}^*(t) = \lambda S_1(t) + (1 - \lambda) S_2(t)$ and $S_2{}^*(t) = \mu S_1(t) + (1 - \mu) S_2(t)$, where $0 < \mu, \lambda < 1, \mu \neq \lambda$.

Without loss of generality, we assume $S_1{}^*(t) \equiv 0$. Thus, $f(t) \geqslant 0$ for all $t \in [0, 1]$, and $f(t) \neq 0, f(t) \neq S_2{}^*(t), S_2{}^*(t) \neq 0$ a.e. on F.

Let

$$\mathscr{S}_{[0,\xi_k]} = \{S: S \in \mathscr{S}_{n,r}, S(t) \equiv 0, t \geqslant \xi_k\},$$

$$\mathscr{S}_{[\xi_k,1]} = \{S: S \in \mathscr{S}_{n,r}, S(t) \equiv 0, t \leqslant \xi_k\},$$

$$\mathscr{S}_{[\xi_k,\xi_\ell]} = \{S: S \in \mathscr{S}_{n,r}, S(t) \equiv 0, t \leqslant \xi_k, t \geqslant \xi_\ell\}.$$

Since 0 is a best one-sided L^1 approximation to $f \in C^1[0, 1]$ from below, our problem reduces to proving that it is the unique best one-sided approximation from $\mathscr{S}_{[\xi_{k_{2i}},\xi_{k_{2i+1}}]}$, $i = 0, 1,..., p$.

Thus, it suffices to consider four cases:

(a) $S_2{}^*(t) \neq 0$ a.e. on $[0, 1]$.

(b) $S_2{}^*(t) \in \mathscr{S}_{[0,\xi_k]}$, $S_2{}^*(t) \neq 0$ a.e. on $[0, \xi_k]$, $1 \leqslant k \leqslant r$.

(c) $S_2{}^*(t) \in \mathscr{S}_{[\xi_k,1]}$, $S_2{}^*(t) \neq 0$ a.e. on $[\xi_k, 1]$, $1 \leqslant k \leqslant r$

(d) $S_2{}^*(t) \in \mathscr{S}_{[\xi_k,\xi_\ell]}$ $S_2{}^*(t) \neq 0$ a.e. on $[\xi_k, \xi_\ell]$, $1 \leqslant k < l \leqslant r$.

Since cases (b) and (c) are equivalent by a change of variable, we prove the theorem for the cases (a), (b), and (d). The above analysis is due to Galkin [2].

The following lemma will be used in the proof of (a). We shall indicate its extensions on considering (b) and (d).

For ease of notation, we define

DEFINITION. *A vector* $\mathbf{x} = (x_1, \ldots, x_\ell)$, $0 \leqslant x_1 < \cdots < x_\ell \leqslant 1$ *is said to have index* $I(\mathbf{x})$, *where* $I(\mathbf{x})$ *counts the number of points in* $\mathbf{x}$, *under the special convention that the endpoints are counted as* $\frac{1}{2}$ *and the interior points are counted as* 1.

Thus, for example, $I((0, x, y)) = 2\frac{1}{2}$, where $0 < x < y < 1$, and $I((0, x, 1)) = 2$ if $0 < x < 1$.

LEMMA A. *Let* $f \in C[0, 1]$, *and* $S^* \in \mathscr{S}_{n,r}$ *be a best one-sided* L^1 *approximation to* $f(t)$. *Then the distinct points of contact of* $f(t)$ *and* $S^*(t)$ *in* $[0, 1]$ *must be of index* $\geqslant (n + r)/2$.

Proof. Assume that the index of the distinct points of contact of $f(t)$ and $S^*(t)$ is $m \leqslant (n + r - 1)/2$. Let $\mathbf{x} = (x_1, \ldots, x_\ell)$, $0 \leqslant x_1 < \cdots < x_\ell \leqslant 1$ denote these points of contact.

Since $2m \leqslant n + r - 1$, there exists an $\tilde{S} \in \mathscr{S}_{n,r}$ satisfying

(1) $\tilde{S}(x_i) = \tilde{S}'(x_i) = 0, \quad$ if $x_i \in (0, 1)$,
$\tilde{S}(0) = 0, \quad$ if $x_1 = 0$,
$\tilde{S}(1) = 0, \quad$ if $x_\ell = 1$;

(2) $\tilde{S}(t) \geqslant 0, \quad t \in [0, 1]$;

(3) $\int_0^1 \tilde{S}(t)\, w(t)\, dt > 0$.

The construction of such a spline is made obvious by the fact that $\{1, t, \ldots, t^{n-1}, (t - \xi_1)_+^{n-1}, \ldots, (t - \xi_r)_+^{n-1}\}$ constitutes a weak Tchebycheff system on $[0, 1]$ and by the technique of smoothing (see [4]).

Let $S_\epsilon(t) = \tilde{S}(t) - \epsilon$, where $\epsilon > 0$ and sufficiently small such that

$$(3^*) \qquad \int_0^1 S_\epsilon(t)\, w(t)\, dt > 0.$$

A contradiction of the optimality of $S^*(t)$ shall now ensue on considering $S^*(t) + \eta S_\epsilon(t)$, for some $\eta > 0$ and small, as follows.

If $S_\epsilon(t) \leqslant 0$, then $f(t) - S^*(t) - \eta S_\epsilon(t) \geqslant f(t) - S^*(t) \geqslant 0$. Let $K = \{t : S_\epsilon(t) \geqslant 0\}$. Since $S_\epsilon(t) \geqslant 0$, $\tilde{S}(t) \geqslant \epsilon > 0$, and by the construction of $\tilde{S}(t)$, $f(t) - S^*(t) \geqslant d > 0$ on K. Choosing $\eta > 0$, sufficiently small, $f(t) -$

$S^*(t) - \eta S_\epsilon(t) \geqslant 0$ for $t \in K$. Thus $f(t) - S^*(t) - \eta S_\epsilon(t) \geqslant 0$ for all $t \in [0, 1]$. From (3*),

$$\int_0^1 (S^*(t) + \eta S_\epsilon(t))\, w(t)\, dt > \int_0^1 S^*(t)\, w(t)\, dt,$$

contradicting the definition of $S^*(t)$. Q.E.D.

Proof of theorem (continued).

Case (a). $S_2{}^*(t) \neq 0$ a.e. on $[0, 1]$.

Since 0 is also a best approximation, so is $\lambda S_2{}^*(t)$ for all $0 \leqslant \lambda \leqslant 1$. From Lemma A there exist ℓ distinct points, of index at least $(n + r)/2$, for which $\lambda S_2{}^*(t) = f(t)$, $0 \leqslant \lambda \leqslant 1$. For $0 < \lambda < 1$, if $\lambda S_2{}^*(t) = f(t)$, then $f(t) = 0$ since $f(t) \geqslant 0$ and $f(t) \geqslant S_2{}^*(t)$. Thus there exist ℓ distinct points, of index at least $(n + r)/2$, for which $S_2{}^*(t) = f(t) = 0$. Recall that $f \in C^1[0, 1]$, and hence if $f(t) = 0$, $t \in (0, 1)$, then $f'(t) = 0$. Because $S_2{}^*(t) \leqslant f(t)$, and $\mathscr{S}_{n,r} \in C^1[0, 1]$ $(n \geqslant 3)$, $S_2^{*\prime}(t) = 0$ at the points of contact described above contained in $(0, 1)$. Therefore $S_2{}^*(t)$ has at least $n + r$ zeros on $[0, 1]$, counting multiplicities.

It easily follows (see [3]) that if $S \in \mathscr{S}_{n,r}$ and $S \neq 0$ a.e. on $[0, 1]$, then S has at most $n + r - 1$ zeros on $[0, 1]$, counting multiplicities, contradiction.

Case (b). $S_2{}^*(t) \in \mathscr{S}_{[0,\xi_k]}$, $S_2{}^*(t) \neq 0$ a.e. on $[0, \xi_k]$, $1 \leqslant k \leqslant r$.

On $[0, \xi_k]$, $S_2{}^*(t)$ is of the form $S_2{}^*(t) = \sum_{i=0}^{n-1} a_i t^i + \sum_{i=1}^{k-1} c_i (t - \xi_i)_+^{n-1}$. $S_2{}^*(t) \neq 0$ a.e. on $[0, \xi_k]$, and therefore has at most $n + k - 2$ distinct zeros in $[0, \xi_k]$, counting multiplicities. However, $S_2^{*(i)}(\xi_k) = 0$, $i = 0, 1,..., n - 2$, and thus $S_2{}^*(t)$ has at most $k - 1$ zeros on $[0, \xi_k)$.

Returning to the analysis of Lemma A, it follows, since $\lambda S_2{}^*(t) \in \mathscr{S}_{[0,\xi_k]}$ is a best approximation to $f(t)$ on $[0, 1]$, that the index of the distinct points of contact of $f(t)$ and $\lambda S_2{}^*(t)$, $0 \leqslant \lambda \leqslant 1$, must be at least $k/2$ on $[0, \xi_k)$. $S_2{}^*(t)$ now has at least k zeros on $[0, \xi_k)$, contradicting the earlier bound.

Case (d). $S_2{}^*(t) \equiv \mathscr{S}_{[\xi_k,\xi_\ell]}$, $1 < k < \ell \leqslant r$, and $S_2{}^*(t) \neq 0$ a.e. on $[\xi_k, \xi_l]$.

Since $S_2{}^*(t) \neq 0$ a.e. on $[\xi_k, \xi_\ell]$, and $S_2^{*(i)}(\xi_k) = S_2^{*(i)}(\xi_l) = 0$, $i = 0, 1,..., n - 2$, it follows that $\ell - k \geqslant n$, and $S_2{}^*(t)$ has at most $\ell - k - n$ zeros in (ξ_k, ξ_l). From the analysis of Lemma A, we obtain a lower bound of $(\ell - k - n + 1)/2$ for the index of the points of contact of $f(t)$ and $\lambda S_2{}^*(t)$ on (ξ_k, ξ_l). Therefore $S_2{}^*(t)$ has at least $\ell - k - n + 1$ zeros in (ξ_k, ξ_l), contradicting our earlier bound. Q.E.D.

Remark 1. We can extrapolate from the above proof necessary conditions for $S^* \in \mathscr{S}_{n,r}$ to be the best one-sided approximation to $f \in C^1[0, 1]$ in terms of the number of zeros of $f(t) - S^*(t)$ on $[\xi_k, \xi_\ell]$, $0 \leqslant k < \ell \leqslant r + 1$.

Remark 2. In contrast to the paper of Galkin [2], we made no overt use of the meshing of knots and zeros of a spline necessary to ensure unique interpolation. This meshing is, in fact, also unnecessary in the proof of Galkin's result.

Remark 3. The above results extend to Tchebycheffian spline functions (see [3]).

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